BROWN-HET 1567
arXiv:0810.4548

N-body Dynamics of Giant Magnons in $\mathbb{R} \times S^2$

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October 26, 2008

Abstract

We pursue the question of multi-magnon dynamics, focusing on the simplest case of magnons moving on $\mathbb{R} \times S^2$ and working at the semiclassical level. Through a Pohlmeyer reduction, the problem reduces to another well known integrable field theory, the sine-Gordon model, which can be exactly described through an N-body model of Calogero type. The two theories coincide at the level of equations of motion, but physical quantities like the energies (of magnons and solitons) and the associated phase shifts are different. We start from the equivalence of the two systems at the level of equations of motion and require that the new (string theory) model reproduces the correct magnon energies and the phase shift, both of which differ from the soliton case. From the comparison of energies we suggest a Hamiltonian, and from requiring the correct phase shift we are led to a nontrivial Poisson structure representing the magnons.

1 Introduction

Recent progress in understanding the Gauge/String duality in $N = 4$ Super Yang-Mills theory resulted in complete specification of the worldsheet S-matrix and the associated spectrum [1–13]. The conjectured exact result received impressive confirmation in both weak coupling Yang-Mills theory calculations and also semiclassical string theory calculations at strong coupling [14–23]. These successes were accomplished due to the integrability property characterizing the string dynamics and present in Yang-Mills theory through its spin-chain representation [1, 24–28]. At the spectrum level there is a complete classification of states in terms of magnon excitations. Their dispersion formula is again known from both weak and strong coupling studies [29, 30].

Even though all orders results have been accomplished, further study of the models and of their integrability structures is still desirable. For instance the spin chain Hamiltonian is reliably known only from weak coupling calculations, its comparison (and agreement) with the string theory Hamiltonian is to some degree purely accidental. In this work we will pursue the question of multi-magnon dynamics. Magnons scatter from each other with known computable phase shifts [3, 20, 31–36] and it is of relevance to determine their interactions. We will do that in the simplest case of magnons moving on $\mathbb{R} \times S^2$ working at the the semiclassical level. The equations of motion in this case (in a timelike-conformal gauge) coincide with those of the $O(3)$ nonlinear sigma model. Multi-magnon solutions have been constructed in these case using several different techniques, involving the dressing and the inverse scattering methods [31, 37–40].

One particular approach (Pohlmeyer reduction method) reduces the problem at the equation of motion level to a well known integrable field theory, the sine-Gordon model [41]. In this reduction the role of magnons is played by sine-Gordon solitons. Much is known about inter-soliton dynamics in the sine-Gordon theory [42]. In particular it can be exactly described through an N-body model generalizing the Calogero-Moser model [43, 44]. The relativistic Ruijsenaars-Schneider model [45] is completely integrable, it summarizes the N-soliton (and anti-soliton) dynamics for a given coupling and can be directly deduced from sine-Gordon theory itself [46]. In turn it can be used as a full dynamical theory, even at the quantum level [47]. It is our goal to establish a related dynamical description for string theory magnons.

The connection between string dynamics and sine-Gordon theory, is known to be highly nontrivial. The two theories coincide at the level of equations of motion, but that is where the comparison stops [30]. Physical quantities like the energies (of magnons and solitons) and the associated phase shifts are different

and it is our intention to clarify somewhat this nontrivial relationship. The nontrivial dynamical connection between the two systems can be traced back to a (nonabelian) dual description of sigma models and the fact that it is in the dual formulation that the connection can be described in canonical terms. This was established in several works by Mikhailov [48–50] and remains to be pursued at the quantum level.

For the question of formulating the dynamical system describing multi-magnon dynamics we start from the fact that at the level of equations of motion it coincides with the soliton or rather the N-body RS model. We then require a further fact, namely that the string theory model ought to reproduce the correct magnon energies and the phase shifts, both of which differ from the soliton case. From the comparison of energies we suggest a Hamiltonian, as the $n = -1$ member of the infinite Hamiltonian sequence [51–53]. Requiring the correct phase shift we are led to a nontrivial Poisson structure representing the N- magnon dynamics.

The content of the paper is as follows. In Section 2 we give a summary of various classical magnon results in $\mathbb{R}\times S^2$ and comparison with sine-Gordon solitons. In Section 3 we review the integrable dynamics of solitons in terms of the N-particle R-S description. In Section 4 we consider the analogous representation for magnons. From comparison of energy eigenvalues we are led to an N-body Hamiltonian given by the inverse of the lax matrix of the RS model. Elaborating on the phase shift we are led to suggest a need for an alternative symplectic form. This symplectic form is explicitly given in the limit of well separated magnons in Section 5. Section 6 is reserved for the conclusions.

2 Semiclassical Giant Magnons in $\mathbb{R} \times S^2$

String dynamics in the $AdS_5 \times S^5$ space-time can be described by the $\sigma-$model action

$$S = \frac{\sqrt{\lambda}}{2\pi} \int \mathrm{d}\tau \, \mathrm{dx} \left\{ \underbrace{\eta^{ab}\partial_a Y^\mu \partial_b Y_\mu + \alpha_1 \left(Y^2 + 1\right)}_{AdS_5} + \underbrace{\eta^{ab}\partial_a X^i \partial_b X_i + \alpha_2 \left(X^2 - 1\right)}_{S^5} \right\},$$

where one embeds both the sphere as the AdS space in $\mathbb{R}^6$ with the respective constraints. This action has a symmetry under $\mathfrak{su}(2,2) \times \mathfrak{so}(6)$. To consider magnons moving on the sphere one restricts the space-time to $\mathbb{R} \times S^5$, with $\mathbb{R}$ being one of the time directions of the AdS_5 space, the respective charges are the generators of rotations in S^5

$$J_{ij} = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} dx \left(X_i \dot{X}_j - X_j \dot{X}_i \right),$$

and the generator of time translations

$$\Delta = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} dx \dot{Y}^0.$$

One considers the limit when the angular momentum $J = J_{12}$ in the direction $\varphi \equiv (12)$ of S^5 is very large, and look at states with $\Delta - J$ finite. The momentum of the excitation p is also kept fixed.

In this limit, we find that the string ground state has $\Delta - J = 0$, which consists of a point-particle with a light-like trajectory along the direction φ, time coordinate $Y^0 \equiv t$ obeying $\varphi - t =$ constant, and sitting at the origin of the spatial directions of AdS_5.

To find excitations above this ground state one looks at solutions rotating in the $Z_1 = X^1 + iX^2$ plane. The remaining four directions of the embedding space we call $\vec{X}$, and $Y^0 \equiv t$ is the time co-ordinate (ultimately from AdS). So the motion is all in the time direction of AdS space, and on the subspace $S^2 \subset S^5$: $\mathbb{R} \times S^2$. By choosing a time-like $t = \tau$, conformal gauge (the induced metric is proportional to the standard metric, $\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \propto \eta_{ab}$) we are looking for solutions that solve the Virasoro constraints

$$\left(\partial_\tau X^i\right)^2 + \left(\partial_x X^i\right)^2 = 1, \qquad \partial_\tau X^i \partial_x X^i = 0,$$

and obey the conformal equations of motion

$$\left(-\partial_\tau^2 + \partial_x^2\right) X^i + X^i \left(-(\partial_\tau X^j)^2 + (\partial_x X^j)^2\right) = 0.$$

Solving the equations Hofman and Maldacena [30] found the Giant Magnon solution:

$$
\begin{aligned}
Y^0 &= \tau, \\
Z_1 &= e^{i\tau}\left(c + i\sqrt{1-c^2}\tanh u\right), \\
\vec{X} &= \vec{n}\sqrt{1-c^2}\mathrm{sech} u,
\end{aligned}
$$

where $c = \cos(p/2)$ is the worldsheet velocity, and (u, v) are boosted worldsheet co-ordinates

$$
\begin{aligned}
&u = \gamma(x - c\tau), \\
&v = \gamma(\tau - cx), \qquad \text{with } \gamma = \frac{1}{\sqrt{1-c^2}} = \frac{1}{\sin(p/2)}.
\end{aligned}
$$

This is a rigidly rotating string along the equator of S^2, with cusps touching this equator and moving at the speed of light.

In the conformal, time-light gauge we started from, the relevant charges can be written as

$$
\begin{aligned}
&\Delta = \frac{\sqrt{\lambda}}{2\pi}\int dx\, 1 && \text{time-translations,} \\
&J = \frac{\sqrt{\lambda}}{2\pi}\int dx\, \mathrm{Im}\left(\overline{Z}_1\partial_t Z_1\right) && \text{angular momentum in } Z_1 \text{ plane,} \\
&p = \frac{1}{i}\int dx \frac{d}{dx}\ln Z_1 && \text{worldsheet momentum.}
\end{aligned}
$$

For the case of the Giant Magnon, Δ and J are infinite, with

$$
E_{mag} = \Delta - J = \frac{\sqrt{\lambda}}{\pi}\sin(p/2). \tag{1}
$$

It is well-known that the theory of classical strings moving on $\mathbb{R}\times S^2$ is related to the sine-Gordon model at the level of the equations of motion. For a conformal gauge solution with $Y^0 = t$, one has the Pohlmeyer [41] identification of a scalar field $\alpha(x,t)$as

$$
\cos\alpha = -\partial_\tau X^i\partial_\tau X^i + \partial_x X^i\partial_x X^i
$$

obeying the sine-Gordon equation

$$
-\partial_\tau\partial_\tau\alpha + \partial_x\partial_x\alpha = \sin\alpha\,.
$$

The point particle is mapped to the vacuum $\alpha = 0$, with zero energy, while the giant magnon is mapped to the simple kink [30]

$$
\alpha = 4\arctan\left(e^{-\gamma(x-c\tau)}\right),
$$

whose energy is ($\hat{\theta}$ is the asymptotic rapidity of the sine-Gordon soliton)

$$
\epsilon_{s.g} = \gamma = \frac{1}{\sin(p/2)} = \cosh\hat{\theta}\,. \tag{2}
$$

The comparison between sine-Gordon model and the classical string theory solutions was seen also for another type of "dual" solutions called single spike solutions (see for example [36, 54, 55]).

But the direct connection between these two theories holds only at the level of equations of motion, and it is non-trivial at the canonical level. Several physical properties are different, in particular the energies and the semiclassical phase shifts. In fact the energies shown above in (1) and (2) exibit an inverse relationship

$$E_{magnon} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\gamma} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\epsilon_{s.g}}.$$

This relation can be generalized further for the scattering solution of two magnons. The two-magnon scattering state, obtained via dressing method [39] can can be mapped through Pohlmeyer's reduction to the scattering solutions of two solitons, whose scattering solution can found in [42]. It can be seen that the energy of the two-magnon scattering solution is related to the energy of each of the solitons in the following way

$$E_{2-mag} = \frac{1}{\epsilon_{s.g,1}} + \frac{1}{\epsilon_{s.g,2}}.$$

The scattering phase for two magnons is calculated in a very similar way to the scattering of two sine-Gordon solitons [56,57]. See also [42] for a review on the classical and semiclassical behavior of sine-Gordon solitons. This is not surprising due to the equivalence of the two classical models through Pohmeyer's map [41] (see also [48, 50]). The time-delay and phase shift of scattering of magnons was also studied through Bethe Ansatz techniques [20, 3, 31, 33, 35].

Since the string and the sine-Gordon equations share a common time t, it obviously follows that the time-delay of scattering of giant magnons (on the string worldsheet) and the time delay for the analogous scattering problem of solitons (in sine-Gordon theory) is the same

$$\Delta t_{sg} = \Delta t_{mag} = \frac{2}{m\sinh\theta}\ln\tanh\theta = \Delta\tau,$$

It does not mean however that the scattering phase shifts of the two problems are the same. In fact they differ, due to the difference in energies stated above. One has the well known relation, where the derivative of the phase shift with respect to energy equals the time delay, in the present case :

$$\frac{\partial\delta_{s.g}}{\partial\epsilon_{s.g}} = \Delta\tau = \frac{\partial\delta_{mag}}{\partial E_{mag}} \quad \Rightarrow \quad \delta_{sg} \neq \delta_{mag}.$$

This will imply that a different interaction is responsible for the behaviour in the two cases.

3 Review of sine-Gordon Dynamics

The dynamics of sine-Gordon solitons can be summarized by a relativistic $N-$body model due to Ruijsenaars and Schneider. These class of models [45, 47] represent a relativistic generalization of the Calogero-Moser models [44, 43]. The relation between the field theoretic system of sine-Gordon solitons and the Lax matrix formulation of the Ruijsenaars-Schneider model was also thoroughly discussed in [46]. In this section we will review some of the aspects of this relation and give a summary of the needed notation. For more details and derivations the reader is directed to the original references.

For establishing the N-body description of soliton dynamics one starts with the N-soliton solution, written as:

$$e^{-i\phi} = \frac{\det(1+A)}{\det(1-A)}.$$

where A is a $N \times N$ matrix with components

$$A_{ij} = 2\frac{\sqrt{\mu_i\mu_j}}{\mu_i+\mu_j}\sqrt{X_iX_j}.$$

The μ_i are the rapidities, and the $X_i = a_i e^{2\left(\mu_i z_+ + \mu_i^{-1} z_-\right)}$ are related to the positions of the soliton through the a_i. Here we use light-cone coordinates $z_\pm = x \pm t$, and $\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t)$. Note that for a soliton or anti-soliton, μ is real and a pure imaginary. The breather solution corresponds to a pair of complex conjugated rapidities $(\mu, \overline{\mu})$ and positions $(a, -\overline{a})$.

The sine-Gordon equation can be described by a Hamiltonian system with the canonical symplectic form (π is the conjugate momentum to the s.G field ϕ)

$$\Omega_{sg} = \int \pi \wedge d\phi.$$

This, by direct substitution can be used to deduce the symplectic form of the soliton variables (a_i, μ_i), which can be seen to reduce to the usual symplectic form after a change of variables.

Considering the evolution of the system in terms of the null plane time $z_+ = \tau$ one has

$$A(\sigma, \tau) = e^{\sigma\mu^{-1}} \tilde{A}(\tau)\, e^{\sigma\mu^{-1}},$$

where $[\mu]_{ij} = \mu_{ij} = \mu_i \delta_{ij}$ is the matrix of rapidities, $\tilde{X}_i = a_i e^{\mu_i \tau}$ are the soliton coordinates. The matrix (coordinate)

$$\tilde{A}(\tau) = 2\frac{\sqrt{\mu_i \mu_j}}{\mu_i + \mu_j}\sqrt{\tilde{X}_i \tilde{X}_j}.$$

is then used to reconstruct the Lax matrix of the N-body system. Through diagonalization one has:

$$\begin{aligned} Q &= U^{-1}\tilde{A}U, \\ L &= U^{-1}\mu U. \end{aligned}$$

where $Q = \operatorname{diag}(Q_1, \cdots, Q_N)$ are the eigenvalues of $\tilde{A}$. The N-soliton solution is then written as $e^{-i\phi} = \prod_{i=1}^{N} \frac{1+Q_i}{1-Q_i}$, and the matrix L is the Lax operator, as its time evolution is given by a (Lax) equation

$$\dot{L} \equiv \frac{dL}{d\tau} = [M, L], \quad M = \dot{U}U^{-1}.$$

Consequently, the quantities $H_n = \operatorname{Tr}(L^n) = \sum_{i=1}^{N} \mu_i^n$ are conserved through the evolution of solitons.

Finally, if we define $\rho_i = \dot{Q}_i / Q_i$, and perform the change of variables $(\mu_i, a_i) \to (Q_i, \rho_i)$, which is a symplectic transformation, we find the Lax matrix to have the same form as the original $\tilde{A}$:

$$L = 2\frac{\sqrt{Q_i Q_j}}{Q_i + Q_j}\sqrt{\rho_i \rho_j}. \tag{3}$$

The Poisson brackets of these two variables Q_i, ρ_i are not canonical, so it is convenient to introduce a new set of variables θ_i conjugated to the variables q_i, given by

$$\rho_i = e^{\theta_i} \prod_{k \neq i} \frac{Q_i + Q_k}{Q_k - Q_i}, \; Q_j = i e^{i q_j}\ .$$

In these new variables, the original symplectic form $\int \pi \wedge d\phi$ is simply given by the usual $\int \theta_i dq_i$, which corresponds to the canonical Poisson brackets. From the sequence of conserved quantities, or Hamiltonians, $H_n = \operatorname{Tr}(L^n)$ we are interested in particular in the $H_{\pm 1}$, which are the generators of the evolution in the light cone coordinates $\tau = z_+$ and $\sigma = z_-$. These are given by

$$H_{\pm 1} = \operatorname{Tr}\left(L^{\pm 1}\right) = \sum_j e^{\pm \theta_j} \prod_{k \neq j} \left| \coth\left(\frac{q_j - q_k}{2}\right) \right|.$$

The full Hamiltonian is given by $H = \frac{1}{2}(H_{+1} + H_{-1}) = \operatorname{Tr}\mathcal{L}_{rs}$, where we define

$$\mathcal{L}_{rs} = \frac{1}{2}\left(L + L^{-1}\right). \tag{4}$$

This system corresponds to a particular case of the $N-$particle relativistic Ruijsenaars-Schneider model [45]. The Lax matrix for the general case of RS model was constructed in [45]. This Lax matrix is defined by

$$L_j = V_i C_{ij} V_j, \tag{5}$$

where

$$V_i \equiv e^{\frac{1}{2}\theta_i} \left(\prod_{k \neq i} f\left(q_i - q_k\right) \right)^{1/2} ,$$

and $C_{ij}\left(q\right)$ is directly related to the choice of $f\left(q\right)$. For a family of interaction potentials of the type given below

$$f\left(q\right) = \left[1 + \alpha / \sinh^2\left(\frac{\mu q}{2}\right)\right]^{1/2} , \quad \mu, \alpha \in \left(0, \infty\right), \tag{6}$$

the components C_{ij} are just given by $C_{ij}\left(q\right) = \left[\cosh\left(\frac{\mu q}{2}\right) + ia \sinh\left(\frac{\mu q}{2}\right)\right]^{-1}$, with $\left(1 + a^2\right)^{-1} = \alpha^2$.

The model has an infinite set of commuting conserved charges

$$H_n = \mathrm{Tr}\left(L^n\right) , \; n \in \mathbb{N},$$

with the Hamiltonian (generator of time translations) and momentum (generator of space translations) given as [45]

$$H = \frac{1}{2}\left(H_1 + H_{-1}\right) = mc^2 \sum_{j=1}^{N} \cosh\theta_j \prod_{k \neq j} f\left(q_k - q_j\right), \tag{7}$$

$$P = \frac{1}{2}\left(H_1 - H_{-1}\right) = mc \sum_{j=1}^{N} \sinh\theta_j \prod_{k \neq j} f\left(q_k - q_j\right), \tag{8}$$

where q_i are the positions of the solitons and θ_i the conjugate rapidities. The interaction between solitons is given by the even function $f\left(q_k - q_j\right)$, reducing to $f \equiv 1$ in the free theory. The RS model is relativistic, as the generators ($\mathcal{B}$ is the generator for Boosts)

$$\mathcal{H}_k = \frac{1}{2}\left(H_k + H_{-k}\right), \quad \mathcal{P}_k = \frac{1}{2}\left(H_k - H_{-k}\right), \quad \mathcal{B} = -\frac{1}{c}\sum q_j,$$

obey the two-dimensional Poincaré algebra:

$$\{\mathcal{H}_k, \mathcal{P}_k\} = 0 , \quad \{\mathcal{H}_k, \mathcal{B}\} = \mathcal{P}_k , \quad \{\mathcal{P}_k, \mathcal{B}\} = \mathcal{H}_k . \tag{9}$$

Next let us discuss the question of the time delay (and phase shift) in the particle picture. Considering the two particle case, one goes to the center-of-mass frame, as in [45]:

$$\begin{aligned} s &\equiv q_1 + q_2 \quad , \qquad \varphi \equiv \frac{1}{2}\left(\theta_1 + \theta_2\right), \\ q &\equiv q_1 - q_2 \quad , \qquad \theta \equiv \frac{1}{2}\left(\theta_1 - \theta_2\right), \end{aligned} \tag{10}$$

The Lax matrix (5) and its inverse are then given by

$$L = e^{\varphi} f\left(q\right) \begin{pmatrix} e^{\theta} & C_{12} \\ \bar{C}_{12} & e^{-\theta} \end{pmatrix} \quad ; \qquad L^{-1} = e^{-\varphi} f\left(q\right) \begin{pmatrix} e^{-\theta} & -C_{12} \\ -\bar{C}_{12} & e^{\theta} \end{pmatrix},$$

where C_{ij} is a 2×2 matrix with entries $C_{11} = C_{22} = 1$, and $C_{12} = \bar{C}_{21} = \left[\cosh\left(\frac{\mu}{2}q\right) + ia \sinh\left(\frac{\mu}{2}q\right)\right]^{-1}$. Now it is simple to check that the Hamiltonian (7) becomes

$$H = 2\cosh\varphi \cosh\theta \, f\left(q\right) = \left(\cosh\theta_1 + \cosh\theta_2\right) f\left(q\right). \tag{11}$$

The momentum given by (8) also becomes:

$$P = 2\sinh\varphi \cosh\theta \, f\left(q\right) = \left(\sinh\theta_1 + \sinh\theta_2\right) f\left(q\right). \tag{12}$$

One comment should be made with respect to the interaction potential $f(q) = \left[1 + \alpha/\sinh^2\left(\frac{\mu q}{2}\right)\right]^{1/2}$. Going back to (6) one can see that for $\alpha = 1$ it reduces to the particular case of a repulsive soliton-soliton interaction in the sine-Gordon model, $f_r(q) = \left|\coth\left(\frac{q}{2}\right)\right|$. An extension of the interaction potential to $\alpha = -1$ leads to the attractive case of soliton-anti-soliton interaction of sine-Gordon, where $f_a(q) = \left|\tanh\left(\frac{q}{2}\right)\right|$.

With the Hamiltonian (11) and choosing certain interacting potentials f one can fully recover the properties of the sine-Gordon soliton- (anti)soliton scattering, such as time delay and phase shift. From (12) it is easy to see that the center of mass $P = 0$ corresponds to $\varphi = 0$. Then the center-of-mass Hamiltonian for two particles is:

$$H_{cm} = \cosh\theta\, f(q)\,.$$

Now we have the relation:

$$\dot{q}^2 + f^2(q) = H_{cm}^2.$$

Because H_{cm} is a constant of motion, so is the quantity $\epsilon \equiv H^2 - 1$. Evaluating ϵ asymptotically, when $x \to \infty$, we obtain $\epsilon = \sinh^2\hat{\theta}$, where $\hat{\theta}$ is the asymptotic center-of-mass rapidity. The time delay is then determined by the time taken along a trajectory from $-q$ to q, as $|q| \to \infty$. For the repulsive soliton-soliton case f_r, we get

$$\int_{-q}^{q} \frac{dq}{\sqrt{\epsilon - \operatorname{csch}^2\left(\frac{q}{2}\right)}} = \frac{4}{\sqrt{\epsilon}} \cosh^{-1}\left(\sqrt{\frac{\epsilon}{\epsilon+1}}\cosh\left(\frac{q}{2}\right)\right)\Bigg|_{2\cosh^{-1}\sqrt{\frac{1+\epsilon}{\epsilon}}}^{q} \underset{q\to\infty}{\to} \frac{2q}{\sinh\hat{\theta}} + \frac{1}{\sinh\hat{\theta}}\ln\left(\tanh\hat{\theta}\right).$$

The first term is the time for each of the solitons to go from $-q$ to q if if it was free (no interaction). The second term is in fact the time delay due to having a repulsive interaction, and correctly reproduces the time delay for a soliton-soliton scattering in sine-Gordon theory, obtained through field theoretic methods.

4 An ansatz for the dynamics of a two-magnon system

Our aim is to describes the N-magnon dynamics in Hamiltonian terms. The appropriate dynamical system ought to be such that it reproduces the classical equations of motions, its energy, momentum and finally phase shift in agreement with the known magnon results [30]. We will begin by focusing on the two-magnon interactions.

We know that the sine-Gordon and the magnons have the same classical equations of motion, and as such the time delay for both systems agrees

$$\Delta t_m(E_m) = \Delta t_{sg}(\epsilon_{sg})\big|_{\epsilon_{sG}=\frac{1}{E_m}}$$

but with different energies. This implies different Hamiltonians for the two systems. With the semiclassical phase shift obeying $\frac{\partial\delta(E)}{\partial E} = \Delta t$ one can try to deduce the (Hamiltonian) dynamics directly from the phase shift itself.

For the sine-Gordon system the center-of-mass Hamiltonian is $H_{sg} = \cosh\theta\, f(q) = \epsilon_{sg}$, the equation of motion for the relative position q gives $\dot{q} = \sqrt{\epsilon_{sg}^2 - f(q)^2}$, and the time delay in terms of the energy is simply given by

$$\Delta t_{sg} = \int \frac{dq}{\dot{q}} = \int \frac{dq}{\sqrt{\epsilon^2 - f(q)^2}}\,.$$

and the scattering phase shift of two sine-Gordon solitons is just given by $\delta_{sg}(\epsilon) = \int d\epsilon\, \Delta t_{sg}$, while for the two-magnons

$$\delta_m(E_m) = \int dE_m\, \Delta t_{sg}(\epsilon_{sg})\big|_{\epsilon_{sg}=\frac{1}{E_m}} = \int dE_m\, E_m \int \frac{dq}{f(q)\sqrt{f^{-2}(q) - E_m^2}}\,. \tag{13}$$

In order to determine which Hamiltonian $H_{cm} \equiv E_m$ produces this phase shift we first perform a change of variables, introducing a new coordinate Q through $dQ = \frac{dq}{f(q)}$. Also define $F(Q) = \frac{1}{f(q(Q))}$.

The interaction then follows (soliton-soliton interaction): for $\mathrm{f}(q) = \coth q$ we find $q = \cosh^{-1}\left(e^{Q}\right)$ and $F\left(Q\right) = \sqrt{1-e^{-2Q}}$.

This means that the limit of relative position $q \to \infty$ corresponds to the new relative position doing the same $Q \to \infty$. Also in this limit, we have $f\left(q\right), F\left(Q\right) \to 1$ (the free theory limit).

After this change of variables, we rewrite the phase shift as

$$\delta\left(E_m\right) = \int dE_m \int dQ \sqrt{\frac{E_m^2}{F^2 - E_m^2}} = \int dE_m \int \frac{dQ}{\dot{Q}},$$

and want to find the center-of-mass Hamiltonian $H_{cm} \equiv E_m$ such that

$$\dot{Q} = \frac{\partial H_{cm}}{\partial \alpha} = \sqrt{\frac{F^2 - H_{cm}^2}{H_{cm}^2}}\,, \tag{14}$$

where α is the new relative rapidity, i.e. the conjugate variable to Q. The differential equation above can be solved to give

$$\sqrt{1 - \left(\frac{H_{cm}}{F}\right)^2} = -\frac{\alpha}{F}.$$

This result is only valid for $\alpha < 0$. Squaring this result, we can solve for H_{cm}, and find

$$H_{cm} = \sqrt{1 - \alpha^2 - e^{-2Q}}. \tag{15}$$

This two-body magnon Hamiltonian appears to be of relativistic (Toda) type. It faithfully reproduces the magnon scattering phase shift. But it is not directly recognizable as a known integrable system. Furthermore it is not obvious how to extend it to the N-body case. First one would need to find a two-body Hamiltonian that reduces to H_{cm} in the center-of-mass. In the limit $Q \to \infty$ we have that

$$H = \varepsilon_1 + \varepsilon_2 = \sin\frac{p_1}{2} + \sin\frac{p_2}{2},$$

where $\varepsilon_{1,2} = \sin\frac{p_{1,2}}{2}$ are the energies of each magnon in the free theory. For only one magnon the Hamiltonian would be given by $H = \sqrt{1-\alpha^2} = \sin\frac{p}{2}$, which means that the relation between the rapidity $\alpha < 0$ and the momentum p is $\alpha = -\cos\frac{p}{2} = \cos(\pi + \frac{p}{2})$. These results will hold in the free theory limit for each magnon. Then a good ansatz for the two-body Hamiltonian, which reproduces the correct result for the free limit, would be

$$H_2 = \sqrt{1 - \alpha_1^2 - e^{-2Q}} + \sqrt{1 - \alpha_2^2 - e^{-2Q}}, \tag{16}$$

with $Q = Q_1 - Q_2$, and the momentum of each magnon is given by $\frac{p_i}{2} = \arccos(\alpha_i) - \pi$.

This construction is non-unique because we do not have the expression of the total momentum. As mentioned we also have no information on the integrability properties of this system, which is fulcral to generalise our results to the dynamics of N-magnon solutions. For these reasons we now pursue a different strategy, based on employing the known integrable structure of the RS model, in particular its Lax matrix L. Together with the classical equivalence between sine-Gordon solitons and giant magnons there was evidence that the poles of the S-matrix of scattering magnons were related to a Calogero type system in the non-relativistic limit [12], thus making us believe that the dynamics of magnons are intimately related to the dynamics of solitons in the RS model. In fact one would hope to describe the dynamics of magnons through a Lax pair formulation whose Lax matrix would be directly related to the Lax matrix of the relativistic RS model.

4.1 The N-magnon Hamiltonian

With the motivation for using the RS integrable structure described above we now proceed to the construction of the associated magnon dynamical system. This will involve specifying both the hamiltonian

and the symplectic structure. As we have emphasized before, the energies of the sine-Gordon solitons and the magnons are inverse of each other. This result leads us to the following ansatz for the N-magnon Hamiltonian:

$$H_m = \mathrm{Tr}\left[\mathcal{L}_{rs}^{-1}\right], \tag{17}$$

where $\mathcal{L}_{rs}$ is related to the Lax matrix of the RS model through (4). We will now study this hamiltonian and consider the two-magnon interaction.

Recall that from (4)

$$\mathcal{L}_{rs} = \frac{L+L^{-1}}{2} = \frac{f(q)}{2}\left\{e^{\varphi}\begin{pmatrix} e^{\theta} & C_{12} \\ \bar{C}_{12} & e^{-\theta}\end{pmatrix} + e^{-\varphi}\begin{pmatrix} e^{-\theta} & -C_{12} \\ -\bar{C}_{12} & e^{\theta}\end{pmatrix}\right\}.$$

The RS Hamiltonian (7) is just the trace of the matrix above. This matrix has the following eigenvalues:[1]

$$h_\pm = \frac{f(q)}{2}\left(\cosh(\varphi+\theta) + \cosh(\varphi-\theta) \pm \sqrt{\left(\cosh(\varphi+\theta) - \cosh(\varphi-\theta)\right)^2 + 4\sinh^2\varphi\left(1 - f(q)^{-2}\right)}\right).$$

Then the Hamiltonian for the 2-magnon problem (17) will be just

$$H_m \;=\; h_+^{-1} + h_-^{-1} \;=\; \frac{1}{f(q)}\frac{2\cosh\theta\cosh\varphi}{\cosh^2\theta + f(q)^{-2}\sinh^2\varphi}.$$

Recall that if M is a diagonalizable matrix with $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ the diagonal matrix of eigenvalues, then for a smooth function $g(M)$ the trace of $g(M)$, it will be given by

$$\mathrm{Tr}\left[g(M)\right] = \mathrm{Tr}\left[g(\Lambda)\right] = \sum_{i=1}^{N} g(\lambda_i). \tag{18}$$

It is easy to check that in the free theory ($f(q) \equiv 1$) we have:

$$H_m^{free} = \frac{1}{\cosh\theta_1} + \frac{1}{\cosh\theta_2} = E_{mag,1} + E_{mag,2},$$

which corresponds to the sum of the energy of the two magnons, as expected.

To have an ansatz for the momentum of the N-body magnon problem, we first look at the momentum for the magnons. In [30] we have that for one magnon the relation between the momenta p_m and the rapidity θ is given by $\cosh\theta = \left[\sin\frac{p_m}{2}\right]^{-1}$. But we know that for the sine-Gordon model the total momentum is $P = \sum_i p_i = \sum_i \sinh\theta_i$. Then a simple comparison allows us to conclude that the momenta for each magnon $p_{m,i}$ is related to the momenta of each soliton p_i by:

$$\sin\left(\frac{p_{m,i}}{2}\right) = \frac{1}{\sqrt{1+p_i^2}}. \tag{19}$$

Thus, a good ansatz for the momenta of the magnon $P_m = \sum_i p_{m,i}$ will be

$$P_m = 2\,\mathrm{Tr}\left[\arcsin\left(\mathbf{1} + \mathcal{P}_{rs}^2\right)^{-1/2}\right], \tag{20}$$

where we defined the momentum matrix for the RS model (whose trace gives the RS momenta given by (8)) to be $\mathcal{P}_{rs} = \frac{L-L^{-1}}{2}$. By knowing the eigenvalues of this last matrix, we can determine P_m by using the result (18):

$$P_m = 2\sum_{i=\pm}\arcsin\left(\frac{1}{\sqrt{1+p_i^2}}\right). \tag{21}$$

[1] For a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, its eigenvalues are simply given by $\lambda_\pm = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$.

The eigenvalues of $\mathcal{P}_{rs}$ are

$$p_{\pm} = \frac{f(q)}{2}\left\{\sinh\theta_1 + \sinh\theta_2 \pm \sqrt{(\sinh\theta_1 - \sinh\theta_2)^2 + 4\cosh^2\varphi\left(1 - f(q)^{-2}\right)}\right\}.$$

The magnon momentum will then be given by

$$\sin\frac{P_m}{2} = \frac{p_+ + p_-}{\sqrt{1+p_+^2}\sqrt{1+p_-^2}}.$$

In the limit of the free theory $f(q) \to 1$, we find the expected relation $P_m^{free} = p_1^m + p_2^m$, i.e. the magnon momentum is the sum of the momenta for each magnon.

Note that because all integrals of motion H_k Poisson commute with each other, so will H_m and P_m:

$$\{H_m, P_m\} = 0.$$

The center of mass condition is given by

$$P_m = 0 \;\Rightarrow\quad \sin\frac{P_m}{2} = 0 \;\Rightarrow\quad \theta_1 + \theta_2 = 0.$$

In the center of mass, the Hamiltonian is simply

$$H_m = \frac{1}{f(q)}(\operatorname{sech}\theta_1 + \operatorname{sech}\theta_2) = \frac{2}{f(q)}\operatorname{sech}\theta,$$

with $f(q)$ the same as before.

A check on our ansatz is determining the classical and semiclassical behaviors of our system, such as the time delay and phase shift for this two-body problem of scattering magnons, and compare them to the known results [30].

We start from the center-of-mass Hamiltonian determined above, and determine the classical equations of motion and time delay. But to do so, we need to choose a Poisson structure. Let us assume that the Poisson structure is the symplectic one. Then the equation of motion for q is just

$$\dot{q} \equiv \frac{\partial H}{\partial\theta} = -H_m \tanh\theta = H_m\sqrt{1 - \frac{1}{4}f(q)^2 H_m^2}.$$

The Hamiltonian is a conserved quantity, $H_m \equiv E$ and can be evaluated when $q \to \infty$, giving $E = \operatorname{sech}\hat{\theta}$, where $\hat{\theta}$ is the asymptotic rapidity. We find the time delay in this case to be

$$\Delta T_m = \int \frac{dq}{\dot{q}} = \cosh^2\hat{\theta}\,\Delta T_{RS}.$$

But this time delay is not correct, nor does it reproduce the right phase shift.

The phase shift is determined by WKB semiclassical methods to be given by the symplectic structure ω (the inverse of the Poisson structure). In phase space variables (x_i, p_i) we have:

$$\delta(E) \equiv \int p_i\,\omega_{ij} dx_j. \tag{22}$$

For the results in this section we have used a canonical Poisson brackets (the standard symplectic structure $\{p_i, x_j\} = \delta_{ij}$), so the phase shift is simply $\delta(E) = \int\theta\,dq$. By solving $H_m \equiv \frac{2}{f(q)}\operatorname{sech}\theta = E$, with respect to the rapidity, $\theta = \cosh^{-1}\left(\frac{2}{f(q)E}\right)$, we can determine the semiclassical phase shift to be

$$\delta(E) = \int \cosh^{-1}\left(\frac{2}{f(q)E}\right) dq. = \int dE\,\Delta T_m.$$

We find that even though our ansatz correctly reproduces the energies and momenta of the magnon system, it does not give the expected classical behaviour (time delay or equations of motion) nor the semiclassical phase shift. But in these calculations we have assumed the usual canonical Poisson brackets, which is equivalent to having a canonical symplectic form for q and p and which resulted in the usual form of the Hamilton-Jacobi equations, namely $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial p}$. One can trace the difference in phase shifts to the different Poisson structures in the two cases. The semiclassical phase shift can be related to the symplectic form ω (the inverse of the Poisson structure) as follows :

$$\delta = \int p_i \omega_{ij} dq_j = \int p_i \omega_{ij} \dot{q}_j dt \,.$$

For the trivial symplectic structure, $\omega \equiv \mathrm{Id}$, and the phase shift has the usual form. But a non-trivial symplectic form is required for reproducing the correct phase shifts and for defining the full magnon N-body dynamics. This we will identify in the next section.

5 Poisson Structure for the N-Magnon Dynamics

We have identified above the N-body Hamiltonian for magnons as one member of the RS hierarchy. We have also understood that a new modified Poisson (and symplectic) structure is needed in order to obtain both the correct equations of motion and the correct magnon phase shift. In the present section we will be able to specify the modified symplectic structure in an approximation of well separated magnons. This approximation which we take just for the purpose of simplifying the problem involves a limit of the RS model when the solitons are far away from each other, called the Toda lattice. The relativistic Toda lattice was introduced in [52] as a relativistic version of the regular Toda lattice [58–60]. In these models the study of master symmetries [61–63] and of recursion relations [64, 65, 53] led to the discovery of a sequence of Hamiltonian/Poisson structures that return the same classical equations of motion, result of the existence of a bi-Hamiltonian system [51].

As seen in [52] we obtain the simpler model of relativistic Toda Lattice from the original Ruijsenaars-Schneider model (7) by considering that the particles are very far from each other $q_{i-1} \ll q_i$.[2] This allows us to keep only the nearest neighbour interactions and these interactions become exponential $f(q) = \sqrt{1 + g^2 e^q}$. Note that we are studying the nonperiodic Toda lattice, for which $q_0 = -\infty$ and $q_{N+1} = \infty$.

The Hamiltonian for the relativistic Toda lattice is given by

$$H = \sum_{i=1}^{N} e^{\theta_i} V_i(q_1, ..., q_N) \,. \tag{23}$$

But now the interaction potential is given by nearest neighbour interactions only

$$V_i(q_1, ..., q_N) = f(q_{i-1} - q_i) f(q_i - q_{i+1}) \,, \quad i = 1, ...N. \tag{24}$$

Also, the symplectic form remains

$$\omega = \sum_{i=1}^{N} dq_i \wedge d\theta_i.$$

This system is integrable and has a Lax matrix formulation, inherited from the RS model (up to some similarity transformation) [52, 65, 53, 66, 67]. To write the Lax matrix we introduce the following change of variables

$$a_j = g^2 e^{q_j - q_{j+1} + \theta_j} \frac{f(q_{j-1} - q_j)}{f(q_j - q_{j+1})} \quad ; \quad b_j = e^{\theta_j} \frac{f(q_{j-1} - q_j)}{f(q_j - q_{j+1})} \,, \quad j = 1, ..., N.$$

[2] In fact Ruijsenaars introduced the limit $\varepsilon \to 0$ of the variables

$$q_j^{\varepsilon} \to q_j - 2j \ln \varepsilon \,, \quad j = 1, \cdots, N.$$

Note that $a_0 = a_N = 0$. The Lax matrix is then given by

$$L = \begin{pmatrix} a_1 + b_1 & a_1 & & & \\ a_2 + b_2 & a_2 + b_2 & a_2 & 0 & \\ \vdots & \vdots & \ddots & \ddots & \\ a_{N-1} + b_{N-1} & a_{N-1} + b_{N-1} & \cdots & a_{N-1} + b_{N-1} & a_{N-1} \\ b_N & b_N & \cdots & b_N & b_N \end{pmatrix}.$$

The Hamiltonian $H_1(q,p)$ given in (23) can be written in the new variables:

$$h_1 = \mathrm{Tr} L = \sum_{i=1}^{N-1} a_i + \sum_{i=1}^{N} b_i, \tag{25}$$

The equations of motion in the (q,θ) coordinates are given by

$$\dot{q}_j = e^{\theta_j} V_j \quad ; \quad \dot{\theta}_j = -\sum_k e^{\theta_k} \frac{\partial V_k}{\partial q_j},$$

which can be obtained from the Hamiltonian (23) by using the symplectic Poisson bracket J_0, defined by $\{q_i, p_j\} = \delta_{ij}$. In the (a,b) variables, the symplectic Poisson bracket J_0 becomes a quadratic Poisson bracket π_2:

$$\{a_i, a_{i+1}\} = -a_i a_{i+1}\ , \quad \{a_i, b_i\} = a_i b_i\ , \quad \{a_i, b_{i+1}\} = -a_i b_{i+1}. \tag{26}$$

From this Poisson bracket and the Hamiltonian (25) one obtains the equations of motion in the (a,b) coordinates

$$\dot{a}_j = a_j \left(b_j - b_{j+1} + a_{j-1} - a_{j+1}\right) \quad ; \quad \dot{b}_j = b_j \left(a_{j-1} - a_j\right). \tag{27}$$

The Toda lattice is an integrable model. It also has a bi-Hamiltonian structure, whose properties are summarized in Section A. In order to construct this bi-Hamiltonian structure one needs to identify two Hamiltonian functions h_1, h_2 and two compatible Poisson tensors π_1, π_2 satisfying the same equations of motion, i.e. if $\nabla = \left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{x^M}\right)$ where x^i are our phase space coordinates, then

$$\frac{d\overline{x}}{dt} = \pi_1 \nabla h_2 = \pi_2 \nabla h_1\,. \tag{28}$$

We already have the Hamiltonian function $h_1 = \mathrm{Tr}L$ (25), and the corresponding quadratic Poisson bracket π_2 (26) such that $\pi_2 \nabla h_1$ gives the equations of motion (27) [65, 53]. A compatible linear Poisson bracket π_1 was found in [68] such that

$$\{a_i, b_i\} = a_i\ ; \quad \{a_i, b_{i+1}\} = -a_i\ ; \quad \{b_i, b_{i+1}\} = a_i\ ,$$

which together with the Hamiltonian $h_2 = \frac{1}{2}\mathrm{Tr}\left(L^2\right)$ also gives the equations of motion (27). These two pairs make a bi-Hamiltonian system with equations of motion given by (28). If we now construct the master symmetries that obey the properties shown in Section A [53, 67], it becomes possible to construct the hierarchy of Poisson brackets with the same equations of motion:[3]

$$\cdots = \pi_0 \nabla h_3 = \pi_1 \nabla h_2 = \pi_2 \nabla h_1 = \pi_? \nabla h_{-1} = \cdots.$$

Our final objective is to make an analogy with the system of N magnons. Recalling the RS Lax matrix $\mathcal{L}_{rs}$, the sine-Gordon model corresponds to the Hamiltonian $H \propto \mathrm{Tr}\mathcal{L}_{rs}$ with the canonical Poisson

[3] The Hamiltonian function h_0 is singular, reason why that point is skipped from the sequence. h_0 can only be defined as a limit

$$h_0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathrm{Tr}\left(L^{\varepsilon}\right) \sim \mathrm{Tr}\left(\ln L\right).$$

But to continue the sequence after this point, one can just do power counting, followed by a direct verification of the equations of motion.

brackets, while the system of magnons was conjecture to correspond its "inverse" $H_m \propto \mathrm{Tr}\mathcal{L}_{rs}^{-1}$, with some other Poisson structure. In the limit we are considering (relativistic Toda), we have

$$H_{sg} \propto \mathrm{Tr}\mathcal{L}_{rs} \to h_1 \quad ; \quad H_{mag} \propto \mathrm{Tr}\mathcal{L}_{rs}^{-1} \to h_{-1}.$$

So, having started from the Hamiltonian $h_1 = \mathrm{Tr}L$, with a quadratic Poisson bracket π_2, we want to find the Poisson bracket corresponding to $h_{-1} = -\mathrm{Tr}\left(L^{-1}\right)$ that gives origin to the equations of motion (27), i.e.

$$\pi_2 \nabla h_1 = \pi_m \nabla h_{-1}.$$

To do so, we will restrict ourselves to $N = 2$.

For the $N = 2$ case, the Lax matrix reduces to

$$L = \begin{pmatrix} a_1 + b_1 & a_1 \\ b_2 & b_2 \end{pmatrix},$$

the Hamiltonian functions are given by

$$\begin{aligned} h_1 &= \mathrm{Tr}L = a_1 + b_1 + b_2; \\ h_2 &= \tfrac{1}{2} \quad \mathrm{Tr}L^2 = \frac{1}{2}\left(a_1^2 + 2a_1b_1 + 2a_1b_2 + b_1^2 + b_2^2\right), \end{aligned}$$

and the corresponding Poisson bracket matrices are

$$\pi_1 = \begin{pmatrix} 0 & a_1 & -a_1 \\ -a_1 & 0 & a_1 \\ a_1 & -a_1 & 0 \end{pmatrix} \quad ; \quad \pi_2 = \begin{pmatrix} 0 & a_1b_1 & -a_1b_2 \\ -a_1b_1 & 0 & 0 \\ a_1b_2 & 0 & 0 \end{pmatrix}.$$

The EOM obtained from this bi-Hamiltonian system is

$$\begin{pmatrix} \dot{a_1} \\ \dot{b_1} \\ \dot{b_2} \end{pmatrix} = \pi_1 \nabla h_2 = \pi_2 \nabla h_1 = \begin{pmatrix} a_1\left(b_1 - b_2\right) \\ -a_1b_1 \\ a_1b_2 \end{pmatrix}.$$

The objective is to determine which π_m given origin to the previous EOM with Hamiltonian function

$$h_{-1} = -\mathrm{Tr}L^{-1} = -\frac{1}{b_1b_2}\left(a_1 + b_1 + b_2\right).$$

We want to construct the next Poisson brackets given the master symmetries. The master symmetries X_1 and X_2 were determined in [67], such that: $X_1\left(h_n\right) = (n+1)\,h_{n+1}$ and $X_2\left(h_n\right) = (n+2)\,h_{n+2}$. These are given by

$$\begin{aligned} X_{1,2} \quad &= \quad r_{1,2}^1 \frac{\partial}{\partial a_1} + s_{1,2}^1 \frac{\partial}{\partial b_1} + s_{1,2}^2 \frac{\partial}{\partial b_2}, \\ \text{with} \quad & \quad r_1^1 = a_1^2 + 3a_1b_2\,;\; r_2^1 = a_1\left(a_1^2 + 5a_1b_1 + 4b_1^2 + 2b_1b_2 - b_2^2\right); \\ & \quad s_1^1 = b_1^2 + 2a_1b_1\,;\; s_2^1 = b_1\left(-2a_1^2 - a_1b_1 - 2a_1b_2 + b_1^2\right); \\ & \quad s_1^2 = b_2^2 - a_1b_2 \quad ;\; s_2^2 = b_2\left(2a_1^2 + 3a_1b_1 + 4a_1b_2 + b_2^2\right). \end{aligned}$$

Then the next Poisson Brackets are given by property 4 (recall that the Poisson matrices are anti-

symmetric):[4]

$$\begin{aligned}
\pi_3 &= -\mathcal{L}_{X_1}\pi_2 = \begin{pmatrix} 0 & a_1b_1(a_1+b_1) & -a_1b_2(a_1+b_2) \\ & 0 & -a_1b_1b_2 \\ & & 0 \end{pmatrix}; \\
\pi_4 &= -\frac{1}{2}\mathcal{L}_{X_2}\pi_2 = \begin{pmatrix} 0 & a_1b_1((a_1+b_1)^2+a_1b_2) & -a_1b_2(a_1(a_1+b_1)+2a_1b_2+b_2^2) \\ & 0 & -a_1b_1b_2(a_1+b_1+b_2) \\ & & 0 \end{pmatrix}.
\end{aligned}$$

With these results we can easily see that $\pi_3\nabla h_{-1}$ does not give the right equations of motion, but $\pi_4\nabla h_{-1}$ does. So the Hamiltonian h_{-1} with Poisson bracket π_4 will give the same classical behavior than the Hamiltonian h_1 with Poisson bracket π_2. The hierarchy is given by (the point π_3, h_0 is not defined)

$$\cdots = \pi_0\nabla h_3 = \pi_1\nabla h_2 = \pi_2\nabla h_1 = \pi_4\nabla h_{-1} = \cdots .$$

All of these pairs generate the same equations of motion and time delay. In particular, the Hamiltonian h_{-1} with (quartic) Poisson bracket π_4 will give the same classical behavior than the Hamiltonian h_1 with Poisson bracket π_2. Since we have found that (in the limit of well separated magnons) the Hamiltonian reduces to

$$H_{mag} = \mathrm{Tr}\mathcal{L}_{rs}^{-1} \to h_{-1},$$

it will reproduce the correct equations of motion (the same as the limiting case of sine-Gordon solitons) as long as we use the quartic Poisson structure π_4 defined above.

For a non degenerate Poisson structure, the phase shift is given by the corresponding symplectic form (the inverse of the Poisson tensor). The usual symplectic form is replaced with the following

$$\int p_i\dot{q}_i \to \int p_i\left(\pi^{-1}\right)_{ij}\dot{q}_j.$$

(For a degenerate Poisson structure one has to check this more carefully.) Consequently, if two different systems have the same equations of motion, the different Poisson Structures give origin to different phase shift.

6 Conclusions & Acknowledgments

We have in the present paper considered the question of an N-particle dynamics that would fully describe interacting magnons at the semiclassical level. For this we have specified the interacting hamiltonian as a member of the RS hierarchy. This hamiltonian had the property that it reproduces energies of magnons. We argued that an alternative symplectic form is needed in order to obtain the correct magnon phase shifts. We have considered the question of the modified symplectic form explicitly for the case of well separated magnons. In this limit one had the results of relativistic Toda theory where a sequence of symplectic forms was already established in the literature.

Altogether the new hamiltonian and the modified symplectic form are defined so to reproduce correctly the original classical equations of motion and therefore the time delay. Regarding future interesting problems we mention the following. We have succeeded in establishing the necessary symplectic form in the limit of well separated magnons. For establishing an exact result one will have to give the multi-Poisson structure for the RS model itself. It is likely that this is definitely possible, although technically (and possibly conceptually) challenging. But one can definitely expect that a sequence of symplectic structures always follows for an integrable system. Generalization of the present construction to magnons moving on

[4]To determine the Lie derivative of the 2-tensor π^{ij}, we use the rule for a general tensor

$$\begin{aligned}
\mathcal{L}_X T^{a_1\ldots a_r}{}_{b_1\ldots b_s} &= X^c\nabla_c T^{a_1\ldots a_r}{}_{b_1\ldots b_s} - \nabla_c X^{a_1} T^{c\ldots a_r}{}_{b_1\ldots b_s} - \cdots - \nabla_c X^{a_r} T^{a_1\ldots c}{}_{b_1\ldots b_s} + \\
&\quad + \nabla_{b_1} X^c T^{a_1\ldots a_r}{}_{c\ldots b_s} + \cdots + \nabla_{b_s} X^c T^{a_1\ldots a_r}{}_{b_1\ldots c}.
\end{aligned}$$

higher spheres [69] is also a challenging task. One would also want to define the dynamics in the periodic case appropriate for string motions with finite J [70].

One of us (AJ) would like to acknowledge the pleasant hospitality of L. Feher and and the organizers of the Oetvos Summer School in Budapest where part of this work was done. We also acknowledge discussions with Kewang Jin, Anastasia Volovich and Mark Spradlin on related topics, We are grateful to Jean Avan for his reading of the manuscript and for his constructive suggestions. This work was supported in part by DOE grant DE-FG02-91ER40688- Task A. I. Aniceto was also supported in part by POCI 2010 and FSE, Portugal, through the fellowship SFRH/BD/14351/2003.

A Bi-Hamiltonian structure of Relativistic Toda Lattice [65, 53, 66, 67]

The relativistic Toda lattice is an integrable model, and has a sequence of conserved quantities

$$h_n = \frac{1}{n}\mathrm{Tr}\left(L^n\right).$$

To this sequence we have a corresponding set of Hamiltonian vector fields $\chi_1, ..., \chi_n$, where $\chi_i = [\pi, h_i]$, where π is some Poisson structure and $[,]$ is the Schouten bracket (Lie bracket). Also, we have a hierarchy of Poisson 2-tensors $\pi_1, ..., \pi_n$ (which are polynomial homogeneous of degree n), and a sequence of master symmetries $X_1, ..., X_n$, which obey the following properties (more information on the properties of these entities can be found in [67] and references therein):

1. the π_n tensors are all Poisson structures. The corresponding Poisson brackets are given by

   $$\{f, g\} = \sum_{i,j} \pi^{ij} \frac{\partial f}{\partial x^i} \wedge \frac{\partial g}{\partial x^j}, \ f, g \in C^\infty.$$

   Note that π^{ij} are the matrix elements of the matrix π_n corresponding to this 2-tensor, and $\overline{x} = \left(x^1, ..., x^M\right)$ are the coordinates of the Hilbert space, in our case $(a_1, ..., a_{N-1}, b_1, ..., b_N)$;

2. functions h_n are in involution with all π_m;

3. $X_n\left(h_m\right) = (n+m)\, h_{m+n}$;

4. $\mathcal{L}_{X_n}\left(\pi_m\right) \equiv [X_n, \pi_m] = (m-n-2)\, \pi_{n+m}$, where $\mathcal{L}_X$ in the Lie derivative in the direction of the vector$X$;

5. $[X_n, X_m] = (m-n)\, X_{n+m}$;

6. $$\pi_n \nabla h_m = \pi_{n-1} \nabla h_{m+1}, \tag{29}$$

   where π_n now denotes the Poisson matrix of the tensor π_n, and $\nabla = \left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{x^M}\right)$.

It is known [51] that once our system is bi-Hamiltonian, which means that we can identify two Hamiltonian functions h_1, h_2 and two compatible Poisson tensors π_1, π_2 satisfying

$$\pi_1 \nabla h_2 = \pi_2 \nabla h_1,$$

then we can find the whole hierarchy stated above, and the equations of motion are just given by

$$\frac{d\overline{x}}{dt} = \pi_1 \nabla h_2 = \pi_2 \nabla h_1 = \pi_0 \nabla h_3 = \cdots. \tag{30}$$

All of these properties are valid for $m, n > 0$ but can be seen to generalize to negative values as well.

In the case one of the Poisson brackets is symplectic, then one can find a recursion operator which can then be applied to the initial symplectic bracket to determine the hierarchy [64, 53]. In our case, we will see that in the (a, b) coordinates the Poisson brackets are not symplectic (not even non degenerate, as we don't have the same number of $a's$ and $b's$) and it is non-trivial to find an extra Poisson bracket in the (p, q) coordinates apart from the symplectic one, in order to form a bi-Hamiltonian system [66]. In this case we will turn to the problem of finding master symmetries that allow us to construct the hierarchy of Poisson structures.

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